Mathematical Notes

Elliptical Integrals

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Elliptical Integrals of I, II and III kinds, Ashgabat - Turkmenistan

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Abstract

Elliptic integrals of 3 kinds were first studied by Euler. We will study them in Legendre canonical forms. Some classic simple results are obtained by Landen's Transformation. In this work we introduce a new, genuine method which leads us to some new formulas that as far as we see cannot be by any means transformed to older ones, which means they are new in their nature. They are more compact and computationally simpler since they involve only polynomial expressions, compared to present ones which include trigonometric and inverse trigonometric expressions.

Chapter 1

Elliptical Integrals of First Kind

1.1 Introduction

Let's consider any function $f(x)$. For any function f we can consider another function g for which $fog(x) = x$, irrespective of its real existence. We now claim that for any function $f(x)$ we may consider the form $fog(x) = h(x)$ [1], just place $h(x)$ instead of x.

Now let's apply these assumptions for calculation of antiderivatives, and see the integral:

$$\int \frac{1}{\sqrt{1-x^2}} dx = f(x) \quad , where \quad f'(x) = \frac{1}{\sqrt{1-x^2}}$$

For function $f'(x)$ we can now consider any form as above [1] $f'(h(x)) = \dfrac{1}{\sqrt{xh'(x)}}$ which is selected for my convinience.

When we equate both $f'(h(x))$ we get:

$$\frac{1}{\sqrt{1-h^2}} = \frac{1}{\sqrt{xh'}}$$

If we solve this differential equation we get:

$$xh' = 1 - h^2, \qquad \frac{h'}{1-h^2} = \frac{1}{x}, \qquad \frac{dh}{1-h^2} = \frac{dx}{x}, \qquad \frac{1}{1-h^2} = \frac{1}{2}\left(\frac{1}{1+h} + \frac{1}{1-h}\right) \tag{1.1}$$

$$\int \frac{1}{2}\left(\frac{1}{1-h} + \frac{1}{1+h}\right) dh = \int \frac{dx}{x} \tag{1.2}$$

$$\frac{1}{2}(ln(1+h) - ln(1-h)) = \frac{1}{2} ln\left(\frac{1+h}{1-h}\right) = lnx$$

$$ln\left(\frac{1+h}{1-h}\right) = lnx^2, \quad \frac{1+h}{1-h} = x^2, \quad \frac{2}{1-h} - 1 = x^2, \quad \frac{2}{1-h} = x^2+1, \quad \frac{2}{x^2+1} = 1-h$$

$$h = 1 - \frac{2}{x^2+1} = \frac{x^2-1}{x^2+1}, \qquad h' = \frac{2 \cdot 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$$\frac{h'}{x} = \frac{4}{(x^2+1)^2}, \qquad \sqrt{\frac{h'}{x}} = \frac{2}{x^2+1} \tag{1.3}$$

If $f'(h) = \dfrac{1}{\sqrt{xh'}}$ then $h'f'(h) = \sqrt{\dfrac{h'}{x}}$. By intergrating both sides we get:

$$f(h) = \int \sqrt{\frac{h'}{x}} dx = \int \frac{2}{(x^2+1)} dx \qquad \text{from} \qquad (1.3) \tag{1.4}$$

So, $f(h) = 2\displaystyle\int \frac{dx}{x^2+1} = 2\arctan x$.

For $h(x)$ let's find $g(x)$ so that $hog = x$

$$hog = 1 - \frac{2}{g^2+1} = x$$

$$1 - x = \frac{2}{g^2+1}, \qquad g^2 + 1 = \frac{2}{1-x}, \qquad g^2 = \frac{2}{1-x} - 1 = \frac{1+x}{1-x}$$

$$g(x) = \pm\sqrt{\frac{1+x}{1-x}}$$

We take $g(x) = \sqrt{\dfrac{1+x}{1-x}}$.

We then find $f(x) = 2\arctan\left(\sqrt{\dfrac{1+x}{1-x}}\right)$. It is easy to verify that derivates of $\arcsin x$ and $f(x)$ are equal. We can also find the constant c for which $f(x)+c = \arcsin x$.

In a similar way we can deal with any other intergral and find the andtiderivative. In next sections we deal with elliptic integrals in the same manner and find their antiderivates. This in its turn leads us to some relations between them with respect to the coefficients k.

1.2 Elliptical Integrals

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{1.5}$$

$$\int \frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-k^2x^2)}} = f(x), \qquad f'(x) = \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \tag{1.6}$$

Let‘s consider the form:

$$f'(h) = \frac{1}{(x(1-k^2x^2)h')^{1/2}} \tag{1.7}$$

$$f'(h) = \frac{1}{((1-h^2)(1-k^2h^2))^{1/2}} = \frac{1}{(x(1-k^2h^2)h')^{1/2}} \tag{1.8}$$

The parts $(1-k^2h^2)^{1/2}$ cancel out

$$\frac{h'}{1-h^2} = \frac{1}{x}, \qquad \frac{dh}{1-h^2} = \frac{1}{1-h^2} = \frac{1}{2}\left(\frac{1}{k-h} + \frac{1}{1+h}\right) \tag{1.9}$$

$$\frac{1}{2}(\ln(1+h) - \ln(1-h)) = \ln x \qquad \ln\left(\frac{1+h}{1-h}\right) = \ln x^2 \tag{1.10}$$

$$1 + x^2 = \frac{-2}{h-2} \qquad h - 1 = \frac{-2}{1+x^2} \qquad h = \frac{-2+1+x^2}{1+x^2} = \frac{x^2-1}{x^2+1} \tag{1.11}$$

$$h = 1 - \frac{2}{1+x^2} \tag{1.12}$$

$$f'(h) = \frac{1}{(x(1-k^2h^2)h)^{1/2}}, \qquad h'f'(h) = \left(\frac{h'}{x(1-k^2h^2)}\right)^{1/2} \tag{1.13}$$

$$f(h) = \int \left(\frac{h'}{x(1-k^2h^2)}\right)^{1/2} dx \tag{1.14}$$

$$\frac{h'}{x} = \frac{\frac{2\cdot 2x}{(1+x^2)^2}}{x} = \frac{4}{(1+x^2)^2}, \qquad \left(\frac{h'}{x}\right)^{1/2} = \frac{2}{1+x^2} \tag{1.15}$$

$$f(h) = \int \left(\frac{h'}{x}\right)^{1/2} \cdot \frac{dx}{(1-k^2h^2)^{1/2}} = \int \frac{2}{1+x^2} \cdot \frac{dx}{(1-k^2h^2)^{1/2}} \tag{1.16}$$

$$1 - \frac{k^2(x^2-1)^2}{(x^2+1)^2} = \frac{(x^2+1)^2 - k^2(x^2-1)^2}{(x^2+1)^2} \tag{1.17}$$

$$h = \frac{x^2-1}{x^2+1}, \qquad f(h) = \int \frac{2}{1+x^2} \cdot \frac{(1+x^2)dx}{((x^2+1) - k^2(x^2-1))^{1/2}} \tag{1.18}$$

$$f(h) = \int \frac{2dx}{((x^2(1-k)+1+k)(x^2(1+k)+1-k))^{1/2}} \tag{1.19}$$

$$f(h) = \int \frac{2dx}{(1-k^2)^{1/2}\left(x^2 + \frac{1+k}{1-k}\right)^{1/2} (1+k)^{1/2}\left(x^2 + \frac{1-k}{1+k}\right)^{1/2}} \tag{1.20}$$

$$f(h) = \int \frac{2dx}{(1-k)^{1/2}\left(x^2 + \dfrac{1+k}{1-k}\right)^{1/2}\left(x^2 + \dfrac{1-k}{1+k}\right)^{1/2}} \tag{1.21}$$

Integral of type $\displaystyle\int \frac{dx}{(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}$ can be transformed to $\displaystyle\frac{1}{a}\int \frac{dt}{(1-t^2)^{1/2}\left(1 - \dfrac{a^2-b^2}{a^2}t^2\right)^{1/2}}$ by substitution:

$x = \dfrac{bt}{\sqrt{1-t^2}}, \quad a^2 = \dfrac{1+k}{1-k}, \quad b^2 = \dfrac{1-k}{1+k}, \quad b^2 = \dfrac{1}{a^2}.$

$$\frac{a^2-b^2}{a^2} = 1 - \frac{1}{a^4} = 1 - \frac{(1-k)^2}{(1+k)^2} = \frac{1+2k+k^2-1+2k-k^2}{(1+k)^2} = \frac{4k}{(1+k)^2}, \quad k' = \frac{2\sqrt{k}}{1+k} \tag{1.22}$$

$$f\left(h\left(\frac{bt}{\sqrt{1-t^2}}\right)\right) = \frac{2}{(1-k)^{1/2}(1+k)^{1/2}} \int \frac{dt}{(1-t^2)^{1/2}(1-(k')^2t^2)^{1/2}} \tag{1.23}$$

Let's define $h\left(\dfrac{bt}{\sqrt{1-t^2}}\right) = \varphi(t)$,

$$f(\varphi(t), k) = \frac{2}{1+k} \cdot f(t, k') \tag{1.24}$$

k, k' - are respective moduli

$$\varphi(t) = \frac{\dfrac{b^2t^2}{1-t^2} - 1}{\dfrac{b^2t^2}{1-t^2} + 1} = \frac{\dfrac{b^2t^2 - 1 - t^2}{1-t^2}}{\dfrac{b^2t^2 + 1 - t^2}{1-t^2}} = \frac{t^2(1+b^2) - 1}{t^2(b^2-1) + 1} \tag{1.25}$$

Let's define

$$F(t, k) = \int_0^t \frac{dx}{\sqrt{1-x^2} \cdot \sqrt{1-k^2x^2}} \tag{1.26}$$

For special values t we have $t = 0, \ \varphi(0) = -1; \quad t = 1, \ \varphi(t) = 1$

$$\int_{-1}^{\varphi(t)} \frac{dx}{\sqrt{1-x^2}\cdot\sqrt{1-k^2x^2}} = \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}\cdot\sqrt{1-k^2x^2}} + \int_{0}^{\varphi(t)} \frac{dx}{\sqrt{1-x^2}\cdot\sqrt{1-k^2x^2}} \tag{1.27}$$

$$= \int_{0}^{1} \frac{dx}{\sqrt{1-x^2}\cdot\sqrt{1-k^2x^2}} + \int_{0}^{\varphi(t)} \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}, \text{ since the integral is even} \tag{1.28}$$

We write $K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$, say $K(k)$ is complete integral.

$$K(k) + F(\varphi(t),k) = f(\varphi(t),k) - f(-1,k) \tag{1.29}$$

Using the identity 1.24 we write

$$K(k) + F(\varphi(t),k) = \frac{2}{1+k}\cdot f(t,k') - \frac{2}{1+k} f(o,k) = \frac{2}{1+k}\cdot F(t,k') \tag{1.30}$$

As a result we obtain a formula:

$$F(\varphi(t),k) = \frac{2}{1+k} F(t,k') - K(k) \tag{1.31}$$

We have a similar formula in Arthur Baker. [p.50]

If we put $t = 1$ we get:

$$K(k) = \frac{2}{1+k} K(k') - K(k),\ \ 2K(k) = \frac{2}{1+k} K(k') \tag{1.32}$$

$$K(k) = \frac{2}{1+k}\cdot K(k') \tag{1.33}$$

Let's transform to form $f(t,k) = \frac{2}{1+k}\cdot f(g(t),k')$ where

$\varphi og = t, \quad g(t) = \sqrt{\frac{1+t}{(1-b^2)t+b^2+1}}$

$$f(t,k) = \frac{2}{1+k}\cdot\frac{2}{1+k_1}\cdots\frac{2}{1+k_n}\cdot f(g_n(t),k_n),\ \ k' = k_1,\ \ k'' = k_2$$

so on

(1.34)

where:

k_n - consecutive application of $\frac{2\sqrt{k}}{1+k}$, $\quad g_n(t) - \underbrace{\text{gogo...g(t)}}_{n\ times}$

Taking limit $lim_{n\to\infty}$:

$$lim_{n\to\infty}\frac{2}{1+k}\cdot\frac{2}{1+k_1}\cdots\frac{2}{1+k_n}=\frac{2}{\pi}K', \quad K'=K(\sqrt{1-k^2})[1][p.45] \tag{1.35}$$

$$lim_{n\to\infty}k_n=1 \quad [1][p.48], \quad lim_{n\to\infty}g_n(t)=u(t) \tag{1.36}$$

$$\begin{aligned} f(t,k)=\frac{2}{\pi}K'f(u(t),1)=\frac{2}{\pi}K'\int\frac{d(u(t))}{1-u^2(t)}=\frac{2K'}{\pi}\frac{1}{2}\ln\left(\frac{1+u(t)}{1-u(t)}\right) \\ f(t,k)=\frac{K'}{\pi}ln\left(\frac{1+u(t)}{1-u(t)}\right) \end{aligned} \tag{1.37}$$

Again:

$$\int_a^b f'(x)dx=f(b,k)-f(a,k) \tag{1.38}$$

$$\begin{aligned} \int_0^t\frac{dt}{(1-t^2)^{1/2}(1-k^2t^2)^{1/2}}=f(t,k)-f(o,k)=\frac{K'}{\pi}\left(\left(\ln\frac{1+u(t)}{1-u(t)}\right)-\ln\left(\frac{1+u(o)}{1-u(o)}\right)\right) \quad OR \\ =\frac{K'}{\pi}\left[\ln\left(\frac{1+g_n(t)}{1-g_n(t)}\right)-\ln\left(\frac{1+g_n(o)}{1-g_n(o)}\right)\right] \quad where \quad n=\infty. \end{aligned} \tag{1.39}$$

We have something similar in Arthur Baker [p.56] which is

$$F(\varphi,k)=\frac{2K'}{\pi}\ln\tan(\frac{\pi}{4}+\frac{1}{2}\varphi_0n)$$

Chapter 2

Elliptical Integrals of Second Kind

Integrals of second kind are represented by

$$\int \sqrt{\frac{1-k^2x^2}{1-x^2}}dx \tag{2.1}$$

We can transform it to

$$\int \frac{1-k^2x^2}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-k^2x^2}}dx = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} - k^2 \int \frac{x^2dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.2}$$

Introduce

$$f(x) = \int \frac{x^2dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad f'(x) = \frac{x^2}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.3}$$

Let's consider the form

$$f'(h) = \frac{h^2}{(x(1-k^2h^2)h')^{1/2}} \tag{2.4}$$

$$f'(h) = \frac{h^2}{((1-h^2)(1-k^2h^2))^{1/2}} = \frac{h^2}{(x(1-k^2h^2)h')^{1/2}} \tag{2.5}$$

$$\frac{h'}{1-h^2} = \frac{1}{x} \qquad h = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} \tag{2.6}$$

$$f(h) = \int h^2 \left(\frac{h'}{x}\right)^{1/2} \cdot \frac{dx}{(1-k^2h^2)^{1/2}}, \qquad \left(\frac{h'}{x}\right)^{1/2} = \frac{2}{1+x^2} \tag{2.7}$$

$$1 - k^2h^2 = \frac{(1-k^2)\left(x^2 + \frac{1+k}{1-k}\right)\left(x^2 + \frac{1-k}{1+k}\right)}{(x^2+1)^2} \tag{2.8}$$

$$f(h) = \int \frac{(x^2-1)^2}{(x^2+1)^2} \cdot \frac{2}{1+x^2} \cdot \frac{(x^2+1)dx}{(1-k^2)^{1/2}(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}, \quad a^2 = \frac{1+k}{1-k}, \quad b^2 = \frac{1-k}{1+k} \tag{2.9}$$

Again by substitution of $x = \frac{bt}{\sqrt{1-t^2}}$, we get:

$$f(h) = \frac{2}{a} \int \frac{h^2}{(1-k^2)^{1/2}} \cdot \frac{dt}{(1-t^2)^{1/2}(1-(k')^2t^2)^{1/2}} \tag{2.10}$$

where $k' = \frac{2\sqrt{k}}{1+k}$, $x = x(t)$

$$f(h) = \frac{2}{(1-k^2)^{1/2}} \cdot \frac{1}{a} \int \frac{h^2 dt}{R'(t)}, \quad R'(t) = (1-t^2)^{1/2}(1-(k')^2t^2)^{1/2} \tag{2.11}$$

Let's calculate $h^2(x(t)) = \varphi^2(t)$

$$\varphi(t) = \frac{t^2(b^2+1)-1}{t^2(b^2-1)+1} = -\frac{1}{k}\left(1 + B \cdot \frac{1}{t^2 - C}\right), \quad B = \frac{1-k^2}{2k}, \quad C = \frac{1+k}{2k} \tag{2.12}$$

$$\varphi^2(t) = \frac{1}{k^2}\left(1 + 2B\frac{1}{t^2-C} + B^2\frac{1}{(t^2-C)^2}\right) \tag{2.13}$$

$$f(h) = \frac{2}{(1-k)^{1/2}(1+k)^{1/2}\frac{(1+k)^{1/2}}{(1-k)^{1}/2}} \int \frac{\varphi^2(t)dt}{R'} \tag{2.14}$$

$$f(h) = \frac{2}{1+k} \cdot \frac{1}{k^2} \int \left(\frac{1}{R'} + 2B \cdot \frac{1}{t^2-C} \cdot \frac{1}{R'} + B^2 \cdot \frac{1}{(t^2-C)^2} \cdot \frac{1}{R'}\right) dt \tag{2.15}$$

Let's transform R':

$$(1-x^2)(1-(k')^2x^2) = 1-(k')^2x^2-x^2+(k')^2x^4 = 1-((k')^2+1)x^2+(k')^2x^4 \quad (2.16)$$

If $R = \sqrt{\alpha+\beta x^2+\gamma x^4}$

$$\frac{xR}{x^2+a} = \alpha_1\int\frac{(x^2+a)dx}{R}+\beta_1\int\frac{dx}{R}+\gamma_1\int\frac{dx}{((x^2+a)R}+\delta_1\int\frac{dx}{(x^2+a)^2R} \quad (2.17)$$

$$\begin{aligned}\alpha_1 = -(2n-5)\gamma, \quad \beta_1 = -(2n-4)(\beta-\delta a\gamma)\\ \gamma_1 = -(2n-3)(3a^2\gamma-2a\beta+\alpha), \quad \delta_1 = (2n-2)(a^3\gamma-a^2\beta+a\alpha)\\ \alpha = 1 \qquad a = -C\\ \beta = -(1+(k')^2) \qquad n = 2 \qquad \gamma = (k')^2\end{aligned}$$

$$\begin{aligned}1\cdot(k')\int\frac{(x^2+a)dx}{R'}-0\cdot\int\frac{dx}{R'}-\underbrace{(3a^2(k')^2+2a(1+(k')^2)+1)}_{A_1}\int\frac{dx}{(x^2+a)R'}\\ +2(\underbrace{a^3(k')^2+a^2(1+(k')^2)+a\cdot 1}_{A_2})\int\frac{dx}{(x^2+a)^2R'} = \frac{xR'}{x^2+a}\end{aligned} \quad (2.18)$$

$$1+(k')^2 = \frac{1+4k}{(1+k)^2} = \frac{1+6k+k^2}{(1+k)^2}$$

$$A_1 = 3\cdot\frac{(1+k)^2}{4k^2}\cdot\frac{4k}{(1+k)^2}-\frac{2(1+k)}{2k}\cdot\frac{1+6k+k^2}{(1+k)^2}+1$$

$$= \frac{3}{k}-\frac{(1+6k+k^2)}{k(1+k)}+1 = \frac{3+3k-1-6k-k^2+k^2+k}{k(1+k)} = \frac{2-2k}{k(1+k)} = \frac{2(1-k)}{k(1+k)}$$

$$A_1 = \frac{2(1-k}{k(1+k)} \quad (2.19)$$

$$A_2 = -\frac{(1+k)}{2k^2}+\frac{1+6k+k^2}{4k^2}-\frac{1+k}{2k} = \frac{-2-2k+1+6k+k^2-2k-2k^2}{4k^2} = \frac{-k^2+2k-1}{4k^2}$$

$$= \frac{-(k^2-2k+1)}{4k^2} = \frac{-(1-2)^2}{4k^2}$$

$$A_2 = \frac{-(1-k)^2}{4k^2} \quad (2.20)$$

From (2.15) we have:

$$f(h)=\frac{2}{1+k}\cdot\frac{1}{k^2}\int\frac{dx}{R'}+\frac{2}{1+k}\cdot\frac{1}{k^2}\cdot 2B\int\frac{dx}{(x^2+a)R'}+\frac{2B^2}{(1+k)}\cdot\frac{1}{k^2}\int\frac{dx}{(x^2+a)^2R'} \tag{2.21}$$

$$B^2=\frac{(1-k)^2(1+k)^2}{4k^2}=-A_2(1+k)^2 \tag{2.22}$$

$$\frac{2B^2}{(1+k)k^2}=\frac{2}{1+k}\cdot\frac{1}{k^2}\cdot -A_2(1+k)^2$$

From (2.18) we have:

$$-2A_2\int\frac{dx}{(x^2+a)^2R'}=-2A_2\cdot\frac{1+k}{k^2}\int\frac{dx}{(x^2+a)^2R'}=\frac{1+k}{k^2}\cdot\left(-2A_2\int\frac{dx}{(x^2+a)^2R'}\right) \tag{2.23}$$

So, from (**??**) :

$$\begin{aligned}\frac{2B^2}{(1+k)k^2}\int\frac{dx}{(x^2+a)^2R'}&=-2A_2\cdot\frac{1+k}{k^2}\int\frac{dx}{(x^2+a)^2R'}=\frac{1+k}{k^2}\cdot\left(-2A_2\int\frac{dx}{(x^2+a)^2R'}\right)\\&=\frac{4}{k(1+k)}\int\frac{x^2dx}{R'}-\frac{2}{k^2}\int\frac{dx}{R'}-\frac{2(1-k)}{k^3}\cdot\int\frac{dx}{(x^2+a)^2R'}-\frac{xR'}{x^2+a}\cdot\frac{1+k}{k^2}\end{aligned} \tag{2.24}$$

Let's sum the corresponding integrals in (2.21): For $\int\frac{dx}{R'}$ the coefficient is: $\frac{2}{1+k}\cdot\frac{1}{k^2}=\frac{2}{k^2}\left(\frac{1}{1+k}-1\right)=\frac{2\cdot -k}{k\cdot(1+k)}=\frac{-2}{k(1+k)}$

For $\int\frac{dx}{(x^2+a)R'}:\frac{4B}{(1+k)k^2}=\frac{4(1-k)(1+k)}{(1+k)\cdot k^2\cdot 2k}=\frac{2(1-k)}{k^3}-\frac{-2(1-k)}{k^3}=0$

which simply vanishes.

So the expression (2.21) takes the form:

$$f(h)=\frac{-2}{k(1+k)}\int\frac{dx}{R'}+\frac{4}{k(1+k)}\int\frac{x^2dx}{R'}-\frac{xR'}{x^2+a}\cdot\frac{1+k}{k^2} \tag{2.25}$$

$$h(x(t))=\varphi(t)=\frac{t^2(b^2+1)-1}{t^2(b^2-1)+1} \tag{2.26}$$

$E(k)$ - for complete integral

For special values

$t = 0 \qquad \varphi = -1$

$t = 1 \qquad \varphi = 1$

1 1

-1 0

$$2E(k) = \frac{-2}{k(1+k)}K(k') + \frac{4}{k(1+k)}E(k') \tag{2.27}$$

$$E(k) = \frac{-1}{k(1+k)}K(k') + \frac{2}{k(1+k)}E(k') \tag{2.28}$$

Let's define:

$$G(k) = K(k) - k^2E(k) = K(k) + \frac{k}{1+k}K(k') - \frac{2k}{1+k}E(k') = K(k') - \frac{2k}{1+k}E(k') \qquad since \qquad K(k) = \frac{1}{1+k}K(k') \tag{2.29}$$

$G(k') = K(k') - (k')^2E(k')$

Let's multiply the above by $\frac{2}{1+k}$:

$$\frac{2}{1+k}G(K) = \frac{2}{1+k}K(k') - (k')^2E(k') = \frac{2}{1+k}K(k') + G(k') - K(k') \tag{2.30}$$

$$since \qquad (k')^2 = \frac{4k}{(1+k)^2}$$

$$\frac{2}{1+k}G(k) = G(k') + (1-k)K(k) \tag{2.31}$$

Define:

$$\int_0^x \frac{t^2dt}{R(t)} = E(x,k)$$

$$\int_0^x \frac{dt}{R(t)} = F(x,k), \qquad R(t) = (1-t^2)^{1/2}(1-k^2t^2)^{1/2}$$

$$G(x,k) = \int_0^x \left(\frac{1-k^2t^2}{1-t^2}\right)^{1/2} dt = \int_0^x \frac{dt}{R(t)} - k^2 \int_0^x \frac{t^2dt}{R(t)} = F(x,k) - k^2E(x,k) \tag{2.32}$$

$$h(x) = \varphi(x) = \frac{x^2(b^2+1)-1}{x^2(b^2-1)+1} \qquad \varphi(0) = -1 \qquad \varphi(1) = 1 \tag{2.33}$$

$$f(\varphi(x),k) - f(-1,k) = \int_{-1}^{\varphi(x)} \frac{t^2dt}{R(t)} \tag{2.34}$$

$$= \int_{-1}^{0} \frac{t^2dt}{R(t)} + \int_0^{\varphi(x)} \frac{t^2dt}{R(t)} = E(k) + E(\varphi(x),k) \tag{2.35}$$

From 2.25 we have

$$E(k) + E(\varphi(x),k) = \frac{-2}{k(1+k)} \int_0^x \frac{dt}{R'(t)} + \frac{4}{k(1+k)} \int_0^x \frac{t^2dt}{R'(t)} - \frac{xR'}{x^2+a} \cdot \frac{1+k}{k^2} \tag{2.36}$$

$$E(\varphi,k) = \frac{-2}{k(1+k)} \cdot F(x,k') + \frac{4}{1+k}E(x,k') - \frac{xR'}{x^2+a} \cdot \frac{1+k}{k^2} - E(k) \tag{2.37}$$

$$G(h,k) = F(h,k) - k^2E(h,k) \tag{2.38}$$

$$= F(h,k) + \underbrace{\frac{2k}{1+k}F(x,k') - \frac{4k}{1+k}E(x,k')} + k^2E(k) + \frac{xR'}{x^2+a} \cdot (1+k)$$

$$\frac{2k}{1+k}F(x,k') + F(x,k')(1+k) - \frac{4k}{(1+k)^2} \cdot (1+k)E(x,k') - (1+k)F(x,k') \tag{2.39}$$

$$(1+k)(F(x,k') - (k')^2E(x,k')) \tag{2.40}$$

$$(1+k)G(x,k') \tag{2.41}$$

$$\frac{2k}{1+k} \cdot F(x,k') - (1+k) \cdot F(x,k') = F(x,k')\left(\frac{2}{1+k)} - (1+k)\right) \tag{2.42}$$

$$= F(x,k')\left(\frac{2k-1-k^2-2k}{1+k}\right)$$

$$= -F(x,k') \cdot \frac{1+k^2}{1+k}$$

$$*F(h,k) = \frac{2}{1+k}F(x,k') - K(k) \tag{2.43}$$

Let's sum the rest of equation:

$$\frac{2}{1+k}F(x,k') - K(k) - F(x,k')\cdot\frac{(1+k^2)}{1+k} + (1+k)\cdot G(x,k') + k^2E(k) + \frac{xR'}{x^3+a}(1+k) \tag{2.44}$$

$$\begin{aligned}
F(x,k')\left(\frac{2-1-k^2}{1+k}\right) &= \frac{1-k^2}{1+k^2} + (1+k)G(x,k') + k^2E(k) + \frac{xR'}{x^3+a}(1+k) \\
G(h,k) &= (1-k)(F(x,k') + (1+k)G(x,k') - (K(k) - k^2E(k)) + \frac{xR'}{x^2+a}(1+k) \\
G(h,k) &= (1+k)G(x,k') + (1-k)\cdot F(x,k') - G(k) + \frac{xR'}{x^2+a}(1+k) \\
where \qquad h(x) = \varphi(x) &= \frac{x^2(b^2+1)-1}{x^2(b^2-1)+1} \\
R'(x) &= ((1-x^2)(1-(k')^2x^2))^{1/2}
\end{aligned} \tag{2.45}$$

we have similar formula in Arthur Baker. [p.99]

Chapter 3

Elliptical Integrals of Third Kind

$$\int \frac{dx}{(1-n^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{3.1}$$

$$-\frac{1}{n^2}\sqrt{(1-x^2)(1-k^2x^2)}, \qquad m=-\frac{1}{n^2}<0$$

$$\int \frac{dx}{(x^2+m)\sqrt{(1-x^2(1-k^2x^2)}} \tag{3.2}$$

$$\int \frac{dx}{(x^2+m)((1-x^2)^{1/2})(1-k^2x^2)^{1/2}} = f(x), \qquad f'(x)=\frac{1}{(x^2+m)(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} \tag{3.3}$$

Let's consider the form:

$$f'(h) = \frac{1}{(h^2+m)[1-(k^2h^2)xh']^{1/2}}$$

$$(3.4)$$

$$f'(h) = \frac{1}{(h^2+m)} \cdot \frac{1}{(1-h^2)^{1/2}} \cdot \frac{1}{(1-k^2h^2)^{1/2}} = \frac{1}{(h^2+m)} \cdot \frac{1}{(1-k^2h^2)^{1/2}} \cdot \frac{1}{(xh')^{1/2}}$$

$$\frac{h'}{1-h^2} = x, \qquad h = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} \tag{3.5}$$

$$f(h) = \int \left(\frac{h'}{x}\right)^{1/2} \cdot \frac{1}{(h^3+m)} \cdot \frac{dx}{(1-k^2h^2)^{1/2}}$$

$$\left(\frac{h'}{x}\right)^{1/2} = \frac{2}{1+x^2}, \qquad 1-k^2h^2 = 1 - \frac{k^2(x^2-1)^2}{(x^2+1)^2} = \frac{(x^2+1)^2 - k^2(x^2-1)^2}{(x^2+1)^2}$$

$$1-k^2h^2 = \frac{(x^2+1-k(x^2-1))(x^2+1+k(x^2-1))}{(x^2+1)^2} = \frac{x^2(1-k)+(1+k)(x^2(1+k)+(1-k))}{(x^2+1)^2}$$

$$= \frac{(1-k^2)\left(x^2+\dfrac{1+k}{1-k}\right)\left(x^2+\dfrac{1-k}{1+k}\right)}{(x^2+1)^2}$$

$$\frac{1}{h^2+m} = \frac{(x^2+1)^2}{m+1} \cdot \frac{1}{x^4+\alpha x^3+1}, \quad \alpha = \frac{2(m-1)}{m+1} \tag{3.6}$$

$$f(h) = \int \frac{2}{1+x^2} \cdot \frac{(x^2+1)^2}{m+1} \cdot \frac{1}{x^4+\alpha x+1} \frac{1}{(1-k^2)^{1/2}} \cdot \frac{(x^2+1)dx}{\left(x^2+\dfrac{1+k}{1-k}\right)^{1/2}\left(x^2+\dfrac{1-k}{1+k}\right)^{1/2}} \tag{3.7}$$

$$\frac{dx}{\left(x^2+\dfrac{1+k}{1-k}\right)^{1/2}\left(x^2+\dfrac{1-k}{1+k}\right)^{1/2}}$$ transforms by substitution:

$$x = \frac{bt}{\sqrt{1-t^2}}, \quad x^2 = \frac{b^2t^2}{1-t^2}, \quad a^2 = \frac{1+k}{1-k}, \quad b^2 = \frac{1-k}{1+k} \quad \textit{to}$$

$$\frac{1}{a} \cdot \frac{dt}{\left(1-\dfrac{a^2-b^2}{a^2}t^2\right)^{1/2}(1-t^2)^{1/2}} = \frac{1}{a} \cdot \frac{1}{(1-t^2)^{1/2}} \cdot \frac{1}{(1-(k')t^2)^{1/2}}, \quad k' = \frac{2\sqrt{k}}{1+k}$$

$$f(h) = \int \frac{2}{m+1} \cdot \frac{(x^2+1)^2}{x^4+\alpha x^2+1} \cdot \frac{1}{(1-k^2)^{1/2}} \cdot \left(\frac{1-k}{1+k}\right)^{1/2} \cdot \frac{dt}{R'(t)} \tag{3.8}$$

$$R'(t) = (1-t^2)^{1/2}(1-(k')^2)t^2)^{1/2} \tag{3.9}$$

$$x(t) = \frac{bt}{\sqrt{1-t^2}} \tag{3.10}$$

$$f(h) = \frac{2}{(m+1)\cdot(k+1)} \int \left(\frac{1}{R'(t)} + \frac{(2-\alpha)x^2}{(x^4+\alpha x^2+1)} \cdot \frac{1}{R'(t)}\right) dt \tag{3.11}$$

Let's consider the second part of paranthesis: $x^2 = z$

$$\frac{z}{z^2+\alpha z+1} = \frac{A}{z-z_1}+\frac{B}{z-z_2}, \qquad A=\frac{z_1}{z_1-z_2}, \qquad B=\frac{z_2}{z_2-z_1}$$

$$z=\frac{-\alpha \pm \frac{4\sqrt{-m}}{m+1}}{2}$$

$$x_1=z_1, \quad x_2=z_2$$

$$z-z_1 = x^2 - x_1 = \frac{b^2t^2}{1-t^2} - x_1 = \frac{b^2t^2 - x_1 + x_1t^2}{1-t^2} = \frac{t^2(b^2+x_1)-x_1}{1-t^2}$$

$$(2-\alpha)\left(\frac{A}{z-z_1}+\frac{B}{z-z_2}\right)\cdot\frac{1}{R'(t)} \tag{3.12}$$

$$\begin{aligned}
\frac{A}{z-z_1}+\frac{B}{z-z_2} &= \frac{A(1-t^2)}{t^2(b^2+x_1)-x_1}+\frac{B(1-t^2)}{t^2(b^2+x_2)-x_2}\\
\frac{1-t^2}{t^2(b^2+x_1)-x_1} &= \frac{-1}{b^2+x_1}+\frac{b^2}{(b^2+x_1)^2}\cdot\frac{1}{t^2-\frac{x_1}{b^2+x_1}}\\
\frac{1-t^2}{t^2(b^2+x_2)-X_2} &= \frac{-1}{b^2+x_2}+\frac{b^2}{(b^2+x_2)^2}\cdot\frac{1}{t^2-\frac{x_2}{b^2+x_2}}\\
\frac{-x_1}{b^2+x-1} &= \eta_1, \qquad \frac{-x_2}{b^2+x_2} = \eta_2
\end{aligned} \tag{3.13}$$

So, $\int\left(\frac{A}{z-z_1}+\frac{B}{z-z_2\cdot R'(t)}\right)dt$ is equal to:

$$\frac{-A}{b^2+x_1}\int\frac{dt}{R'(t)} - B\cdot\frac{1}{b^2+x_2}\int\frac{dt}{R'(t)} + \frac{Ab^2}{(b^2+x_1)^2}\int\frac{1}{t^2+\eta_1}\cdot\frac{dt}{R'(t)} + \frac{Bb^2}{(b^2+x_2)^2}\int\frac{1}{t^2+\eta_2}\cdot\frac{dt}{R'(t)} \tag{3.14}$$

Let's evaluate the inside of paranthesis:

$$\begin{aligned}
&\left(1-(2-\alpha)\left(\frac{A}{62+x_1}+\frac{B}{b^2+x_2}\right)\right)\int\frac{dt}{R'(t)}\\
&+\frac{(2-\alpha)\cdot Ab^2}{(b^2+x_2)^2}\int\frac{dt}{(t^2+\eta_1)(R'(t))}+\frac{(2-\alpha)Bb^2}{(b^2+x_2)^2}\int\frac{dt}{(t^2+\eta_2)(R'(t)}\\
&C_1\int\frac{dt}{R'(t)}+C_2\int\frac{dt}{(t^2+\eta_1)R'(t)}+C_3\int\frac{dt}{(t^2+\eta_2)R'(t)}
\end{aligned} \tag{3.15}$$

C_1, C_2, C_3 - for correspnding coefficients let's change the notation:

$$p(t,m,k) = \int \frac{1}{t^2+m} \cdot \frac{dt}{R(t)} \tag{3.16}$$

$$f(t,k) = \int \frac{dt}{R(t)} \tag{3.17}$$

$$p(\varphi(t),m,k) = M(C_1 f(t,k') + C_2 p(t,n,k') + C_3 p(t,\eta_2,k')) \tag{3.18}$$

$$M = \frac{2}{(m+1)(1+k)} \tag{3.19}$$

Define:

$$\int_0^t \frac{dx}{R(x)} = F(t,k), \qquad \int_0^t \frac{1}{x^2+m} \cdot \frac{dx}{R(x)} = P(t,m,k) \tag{3.20}$$

$$\varphi(t) = \frac{t^2(b^2+1)-1}{t^2(b^2-1)+1}, \qquad so \quad \varphi(0) = -1 \tag{3.21}$$

$\varphi(t)$	t
-1	0

$$\int_{-1}^{\varphi(t)} \frac{1}{t^2+m} \cdot \frac{dt}{R(t)} = M(C_1, F(t,k')) + C_2 P(t,n,k') + C_3 P(t,n,k') \tag{3.22}$$

$$P(\varphi(t),m,k) + P(m,k) = MC_1 F(t,k') + MC_2 P(t,n_1,k') + MC_3 P(t,\eta_2,k') \tag{3.23}$$

where $P(m,k)$ for complete ingtegral

$$P(\varphi(t),m,k) = M_1 C_1 F(t,k') + MC_2 P(t,n_1,k') + MC_3 P(t,\eta_2,k') - P(m,k) \tag{3.24}$$

we have a similar formula in Arthur Baker [p.107]

For special values $\varphi(t) = -1, \quad \varphi(t) = 1$

$$\left.\begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array}\right\} \text{t}$$

$$2P(m,k) = M(C_1, K(k') + C_2 P(\eta_1,k') + C_3 P(\eta_2,k')) \tag{3.25}$$

Bibliography

[1] A.L. Baker. Elliptic Functions: An Elementary Text-book for Students of Mathematics. Cornell University Library historical math monographs. J. Wiley & Sons, 1890.